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Duality Transformations in Electrodynamics

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Electrodynamics admitting a duality transformation group is considered. For such an electrodynamics an extension of the classical Rainich–Misner–Wheeler theory is presented.

1. INTRODUCTION

It is well known (Rainich, 1925; Misner and Wheeler, 1957) that the source-free Einstein–Maxwell equations describing a linear Maxwell electromagnetic field $f_{ij} = -f_{ji}$, $i, j = 1, \ldots, 4$, in a space-time of metric g_{ij} are invariant under the group of *duality rotations* defined by

$$x'^i = x^i, \qquad g'_{ij} = g_{ij}, \qquad f'_{ij} = f_{ij} \cos\varphi + i\, {}^*f_{ij} \sin\varphi, \qquad \varphi \in \mathbb{R} \quad (1.1)$$

where

$${}^*f_{ij} := -\frac{i}{2}\sqrt{-g}\, \epsilon_{ijkl} f^{kl}, \qquad g := \det\|g_{ij}\|, \qquad x^i$$

are local coordinates. In terms of 3-vector fields **E**, **B**, **D**, and **H** (see Section 2) the duality rotation of the electromagnetic field can be written

$$\mathbf{E}' = \mathbf{E}\cos\varphi + \mathbf{H}\sin\varphi, \qquad \mathbf{H}' = -\mathbf{E}\sin\varphi + \mathbf{H}\cos\varphi \qquad (1.2)$$

or

$$\mathbf{D}' = \mathbf{D}\cos\varphi + \mathbf{B}\sin\varphi, \qquad \mathbf{B}' = -\mathbf{D}\sin\varphi + \mathbf{B}\cos\varphi \qquad (1.3)$$

It is also known that the invariance of the source-free Einstein–Maxwell equations under the duality rotation group (1.1) leads to a conserved quantity (Deser and Teitelboim, 1976; Fushchich and Nikitin, 1983;

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Przanowski and Maciołek-Niedźwiecki, 1992). As shown by Deser and Teitelboim (1976), the duality transformation in the sense of (1.1) does not exist in the case of Yang–Mills fields.

On the other hand, it has been shown (Salazar *et al.*, 1987), that one can find a wide class of solutions to the Einstein–Maxwell equations within nonlinear electrodynamics which admits the duality rotation. These solutions generalize the well-known NUT solutions with Λ. Moreover, it also has been demonstrated (Salazar *et al.*, 1989) how the problem concerning the propagation of signals in nonlinear electrodynamics can be considerably simplified if the invariance under the duality rotation is assumed.

In the present paper we consider a natural generalization of the duality rotation group which is called the *duality transformation group* (Sections 2 and 3). Then we find all models of electrodynamics which admit the duality transformation group. In particular, it is shown that for any (linear or nonlinear) electrodynamics which: (i) admits a duality transformation group, (ii) satisfies the dominant energy condition, and, (iii) corresponds to the Maxwell electrodynamics for weak fields, the duality transformation group appears to be exactly the duality rotation group (Section 4). This result has been previously found by Salazar *et al.* (1987). Here it is considered from the general viewpoint. Finally, in Section 5 we give an extension of the classical Rainich–Misner–Wheeler already unified theory (Rainich, 1925; Misner and Wheeler, 1957) to the case of linear or nonlinear electrodynamics satisfying the conditions (i)–(iii).

The problem of a conservation law connected with the duality transformation group in nonlinear electrodynamics will be considered elsewhere.

2. (3 + 1)-FORMALISM IN ELECTRODYNAMICS

We deal with a vacuum electromagnetic field in a space-time M_4 endowed with metric g_{ij}, $i, j = 1, \ldots, 4$, of the signature $(+++-)$ (Born and Infeld, 1934; Plebański, 1968; Białynicki-Birula and Białynicka-Birula, 1975; Salazar *et al.*, 1987). The electromagnetic field is described by the potential A_i and by the antisymmetric tensor p^{ij} ($p^{ij} = -p^{ji}$). Then the Lagrangian of the electromagnetic field is assumed to be of the form

$$L = -\frac{1}{2} p^{ij} f_{ij} + K(P, Q) \tag{2.1}$$

where

$$\begin{gathered} f_{ij} := \partial_i A_j - \partial_j A_i, \qquad \partial_i := \frac{\partial}{\partial x^i}; \qquad P := \frac{1}{4} p_{ij} p^{ij}, \qquad Q := \frac{1}{4} p_{ij} \, {}^*p^{ij} \\ {}^*p_{ij} := -\frac{i}{2} \sqrt{-g}\, \epsilon_{ijkl} p^{kl}, \qquad g := \det\|g_{ij}\| \end{gathered} \tag{2.2}$$

ϵ_{ijkl} is the totally antisymmetric Levi-Civita tensor; the function $K(P, Q)$ is called the *structural function*. The total action for the gravitational electromagnetic field reads

$$S = \int d^4x \sqrt{-g}\left[\frac{1}{16\pi}(R + 2\Lambda) + L\right] \tag{2.3}$$

where R denotes the curvature scalar and Λ is the cosmological constant.

Performing the variation of S with respect to g_{ij}, A_i, and p^{ij} and then using the least action principles $\delta S = 0$, one gets the following set of equations:

Einstein equations

$$R_{ij} - \frac{1}{2} R g_{ij} = -8\pi T_{ij} + \Lambda g_{ij} \tag{2.4}$$

where R_{ij} is the Ricci tensor and T_{ij} is the energy-momentum tensor

$$T_{ij} = p_i^k f_{jk} + L g_{ij} \tag{2.5}$$

Maxwell equations

$$\partial_{[i} f_{jk]} = 0, \qquad p^{ij}{}_{;j} = 0 \tag{2.6}$$

where the square bracket stands for the antisymmetrization and the semicolon denotes the covariant derivative.

"*Material equations*"

$$f_{ij} = \frac{\partial K}{\partial P} p_{ij} + \frac{\partial K}{\partial Q} {}^*p_{ij} \tag{2.7}$$

We now consider a $(3 + 1)$-decomposition of the space-time M_4

$$M_4 = M_3 \times M_1, \qquad \dim M_3 = 3, \qquad \dim M_1 = 1 \tag{2.8}$$

The metric $\gamma_{\mu\nu}$ ($\mu, \nu = 1, 2, 3$) on M_3 is defined by

$$\gamma_{\mu\nu} = g_{\mu\nu} - \frac{g_{4\mu} g_{4\nu}}{g_{44}} \tag{2.9}$$

Define the following 3-vectors (compare with Landau and Lifschitz, 1973)

$$D^\mu = \sqrt{-g_{44}}\, p^{4\mu}, \qquad H_\mu = \sqrt{-g_{44}}\left(\frac{1}{2}\sqrt{\gamma}\,\epsilon_{\mu\nu\sigma} p^{\nu\sigma}\right) \tag{2.10a}$$

$$E_\mu = -f_{4\mu}, \qquad B^\mu = \frac{1}{2\sqrt{\gamma}}\epsilon^{\mu\nu\sigma} f_{\nu\sigma} \tag{2.10b}$$

$\gamma := \det\|\gamma_{\mu\nu}\|$; Greek indices run through 1, 2, 3. Then one finds easily the

Maxwell equations (2.6) in terms of 3-vectors to be of the form

$$\partial_4(\sqrt{\gamma}B^\mu) + \epsilon^{\mu\nu\sigma}\partial_\nu E_\sigma = 0, \qquad \partial_\mu(\sqrt{\gamma}B^\mu) = 0 \tag{2.11a}$$

$$\partial_4(\sqrt{\gamma}D^\mu) - \epsilon^{\mu\nu\sigma}\partial_\nu H_\sigma = 0, \qquad \partial_\mu(\sqrt{\gamma}D^\mu) = 0 \tag{2.11b}$$

Moreover, from (2.7) one gets

$$f_{ij} = 2\frac{\partial K}{\partial p^{ij}}, \qquad {}^*f_{ij} = 2\frac{\partial K}{\partial {}^*p^{ij}} \tag{2.12}$$

Consequently, (2.10a), (2.10b), and (2.12) lead to the following material equations:

$$E_\mu = -\frac{\partial}{\partial D^\mu}(\sqrt{-g_{44}}K), \qquad B^\mu = \frac{\partial}{\partial H_\mu}(\sqrt{-g_{44}}K) \tag{2.13}$$

Simple but tedious manipulations give

$$\begin{aligned} P &= -\frac{1}{2}\mathbf{D}\cdot\mathbf{D} - \frac{1}{2g_{44}}\mathbf{H}\cdot\mathbf{H} + \mathbf{D}\cdot(\mathbf{H}\times\mathbf{n}) - \frac{1}{2}(\mathbf{H}\times\mathbf{n})\cdot(\mathbf{H}\times\mathbf{n}) \\ Q &= \frac{i}{\sqrt{-g_{44}}}\mathbf{D}\cdot\mathbf{H} \end{aligned} \tag{2.14}$$

where the natural 3-vector notation is used, i.e.,

$$\mathbf{D}\cdot\mathbf{D} := \gamma_{\mu\nu}D^\mu D^\nu, \qquad \mathbf{D}\cdot\mathbf{H} := \gamma_{\mu\nu}D^\mu H^\nu, \ldots, \qquad (\mathbf{H}\times\mathbf{n})^\mu := \frac{1}{\sqrt{\gamma}}\epsilon^{\mu\nu\sigma}H_\nu n_\sigma$$

and

$$n_\sigma := -\frac{g_{4\sigma}}{g_{44}}, \qquad \gamma^{\mu\nu}\gamma_{\nu\sigma} = \delta^\nu_\sigma$$

Greek indices are manipulated by $\gamma_{\mu\nu}, \gamma^{\mu\nu}$. (One can easily show that $n^\mu = g^{4\mu}, \gamma^{\mu\nu} = g^{\mu\nu}$.) Analogously we find the invariants

$$\begin{aligned} F &:= \frac{1}{4}f_{ij}f^{ij} = \frac{1}{2g_{44}}\mathbf{E}\cdot\mathbf{E} + \frac{1}{2}\mathbf{B}\cdot\mathbf{B} + \mathbf{B}\cdot(\mathbf{E}\times\mathbf{n}) + \frac{1}{2}(\mathbf{E}\times\mathbf{n})\cdot(\mathbf{E}\times\mathbf{n}) \\ G &:= \frac{1}{4}f_{ij}{}^*f^{ij} = \frac{i}{\sqrt{-g_{44}}}\mathbf{E}\cdot\mathbf{B} \end{aligned} \tag{2.15}$$

Finally, for completeness, we can also write the Maxwell equations (2.11a) and (2.11b) in 3-vector notation (compare with Landau and Lifschitz, 1973) as follows:

$$\frac{1}{\sqrt{\gamma}}\partial_4(\sqrt{\gamma}\mathbf{B}) + \operatorname{rot}\mathbf{E} = 0, \qquad \operatorname{div}\mathbf{B} = 0 \tag{2.16a}$$

$$\frac{1}{\sqrt{\gamma}}\partial_4(\sqrt{\gamma}\mathbf{D}) - \operatorname{rot}\mathbf{H} = 0, \qquad \operatorname{div}\mathbf{D} = 0 \tag{2.16b}$$

where

$$(\operatorname{rot}\mathbf{E})^\mu = \frac{1}{\sqrt{\gamma}}\epsilon^{\mu\nu\sigma}\partial_\nu E_\sigma, \qquad \operatorname{div}\mathbf{B} = \frac{1}{\sqrt{\gamma}}\partial_\mu(\sqrt{\gamma}\mathbf{B}^\mu), \qquad \text{etc.}$$

Then the energy-momentum tensor (2.5) can be found to be

$$T^4{}_4 = -\frac{1}{\sqrt{-g_{44}}}M, \quad T^4{}_\nu = \frac{1}{\sqrt{-g_{44}}}(\mathbf{D}\times\mathbf{B})_\nu, \quad T^\mu{}_4 = \frac{1}{\sqrt{-g_{44}}}(\mathbf{H}\times\mathbf{E})^\mu$$
$$T^\mu{}_\nu = -\frac{1}{\sqrt{-g_{44}}}[D^\mu E_\nu + B^\mu H_\nu - (\mathbf{D}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{H}-M)\delta^\mu_\nu] \tag{2.17}$$

where

$$M := \mathbf{B}\cdot\mathbf{H} - \sqrt{-g_{44}}K \tag{2.18}$$

Now as the second of relations (2.13) gives $\mathbf{H}$ as a function of g_{ij}, $\mathbf{B}$, and $\mathbf{D}$, we will consider M to be a function $M = M(g_{ij}, \mathbf{D}, \mathbf{B})$. Then from (2.13) and (2.18) one gets

$$E_\mu = \frac{\partial M}{\partial D^\mu}, \qquad H_\mu = \frac{\partial M}{\partial B^\mu} \tag{2.19}$$

For the Maxwell equations we get

$$\partial_4(\sqrt{\gamma}B^\mu) + \epsilon^{\mu\nu\sigma}\partial_\nu\left(\frac{\partial M}{\partial D^\sigma}\right) = 0, \qquad \partial_\mu(\sqrt{\gamma}B^\mu) = 0 \tag{2.20a}$$

$$\partial_4(\sqrt{\gamma}D^\mu) - \epsilon^{\mu\nu\sigma}\partial_\nu\left(\frac{\partial M}{\partial B^\sigma}\right) = 0, \qquad \partial_\mu(\sqrt{\gamma}D^\mu) = 0 \tag{2.20b}$$

Note that if one considers $(\sqrt{\gamma}\mathbf{B}, \sqrt{\gamma}\mathbf{D})$ to be the canonical variables (compare with Białynicki-Birula and Białynicka-Birula, 1975), then $\sqrt{\gamma}M$ is the Hamiltonian. We have also $M = \mathbf{D}\cdot\mathbf{E} + \sqrt{-g_{44}}L$.

3. DUALITY TRANSFORMATIONS

We consider a one-parameter local group of transformations G_1,

$$x'^i = x^i, \qquad g'_{ij} = g_{ij}, \qquad \mathbf{D}' = \mathbf{D}'(x^i, \mathbf{D}, \mathbf{B}; \tau), \qquad \mathbf{B}' = \mathbf{B}'(x^i, \mathbf{D}, \mathbf{B}; \tau)$$
$$\tau \in (-\varepsilon, \varepsilon) \subset \mathbb{R}, \qquad \varepsilon > 0 \tag{3.1}$$

leaving the Maxwell equations (2.20a) and (2.20b) and the Einstein equations (2.4) invariant. To find this group we use the jet space formalism

(Ibragimov, 1985; Olver, 1986). The infinitesimal operator of G_1 reads

$$X = \xi^\mu \frac{\partial}{\partial D^\mu} + \eta^\mu \frac{\partial}{\partial B^\mu}$$
$$\xi^\mu = \frac{\partial D'^\mu}{\partial \tau}\bigg|_{\tau=0}, \qquad \eta^\mu = \frac{\partial B'^\mu}{\partial \tau}\bigg|_{\tau=0} \tag{3.2}$$

Then the second prolongation of X takes the form

$$X_2 = X + d_i(\xi^\mu)\cdot\frac{\partial}{\partial D_i^\mu} + d_i(\eta^\mu)\cdot\frac{\partial}{\partial B_i^\mu} + d_i d_j(\xi^\mu)\cdot\frac{\partial}{\partial D_{ij}^\mu} + d_i d_j(\eta^\mu)\cdot\frac{\partial}{\partial B_{ij}^\mu} \tag{3.3}$$

where d_i denotes the total derivative

$$d_i := \partial_i + D_i^\mu \frac{\partial}{\partial D^\mu} + B_i^\mu \frac{\partial}{\partial B^\mu} + \sum_{s\geq 1}\left(D^\mu_{ii_1\ldots i_s}\frac{\partial}{\partial D^\mu_{i_1\ldots i_s}} + B^\mu_{ii_1\ldots i_s}\frac{\partial}{\partial B^\mu_{i_1\ldots i_s}}\right) \tag{3.4}$$

and D_i^μ, B_i^μ, $D^\mu_{ii_1\ldots i_s}$, and $B^\mu_{ii_1\ldots i_s}$ stand for appropriate coordinates in a relevant jet space,

$$D_i^\mu := \partial_i D^\mu, \qquad D^\mu_{ii_1\ldots i_s} := \partial_{ii_1\ldots i_s} D^\mu, \ldots \tag{3.5}$$

Denote the Einstein–Maxwell equations (2.4), (2.20a), and (2.20b) by

$$\mathscr{F} = 0 \tag{3.6}$$

Then the invariance condition of (3.6) under G_1 reads

$$X_2 \mathscr{F}|_{\mathscr{F}=0} = 0 \tag{3.7}$$

where $|_{\mathscr{F}=0}$ means "restricted to $\mathscr{F} = 0$."

The invariance condition (3.7) applied to the second equation of (2.20a) gives

$$[\eta^\mu \partial_\mu] \ln \sqrt{\gamma} + d_\mu(\eta^\mu)]|_{\mathscr{F}=0} = 0 \tag{3.8}$$

Hence

$$\left(\eta^\mu \partial_\mu \ln\sqrt{\gamma} + \partial_\mu \eta^\mu + D_\mu^\nu \frac{\partial \eta^\mu}{\partial D^\nu} + B_\mu^\nu \frac{\partial \eta^\mu}{\partial B^\nu}\right)\bigg|_{\mathscr{F}=0} = 0 \tag{3.9}$$

Then the second equations of (2.20a) and (2.20b) yield

$$\Bigg[\eta^\mu\partial_\mu \ln\sqrt{\gamma} + \partial_\mu\eta^\mu + \sum_{\mu\neq\nu}\left(D_\mu^\nu\frac{\partial\eta^\mu}{\partial D^\nu} + B_\mu^\nu \frac{\partial \eta^\mu}{\partial B^\nu}\right)$$
$$+ D_1^1\frac{\partial\eta^1}{\partial D^1} + D_2^2\frac{\partial\eta^2}{\partial D^2} + (-D^\mu\partial_\mu \ln\sqrt{\gamma} - D_1^1 - D_2^2)\frac{\partial\eta^3}{\partial D^3}$$
$$+ B_1^1\frac{\partial\eta^1}{\partial B^1} + B_2^2\frac{\partial\eta^2}{\partial B^2} + (-B^\mu\partial_\mu \ln\sqrt{\gamma} - B_1^1 - B_2^2)\frac{\partial\eta^3}{\partial B^3}\Bigg]\Bigg|_{\mathscr{F}=0} = 0 \tag{3.10}$$

Therefore

$$\frac{\partial \eta^\mu}{\partial D^\nu} = 0, \qquad \frac{\partial \eta^\mu}{\partial B^\nu} = 0 \qquad \text{for} \quad \mu \neq \nu \tag{3.11a}$$

$$\frac{\partial \eta^1}{\partial D^1} = \frac{\partial \eta^2}{\partial D^2} = \frac{\partial \eta^3}{\partial D^3}, \qquad \frac{\partial \eta^1}{\partial B^1} = \frac{\partial \eta^2}{\partial B^2} = \frac{\partial \eta^3}{\partial B^3} \tag{3.11b}$$

$$\left(\eta^\mu - D^\mu \frac{\partial \eta^3}{\partial D^3} - B^\mu \frac{\partial \eta^3}{\partial B^3}\right) \partial_\mu \ln \sqrt{\gamma} + \partial_\mu \eta^\mu = 0 \tag{3.11c}$$

(Remember that in the present section the partial derivative ∂_μ is considered in the sense of the jet space formalism.) From (3.11a) and (3.11b) one infers that

$$\eta^\mu = f(x^i) D^\mu + h(x^i) B^\mu + l^\mu(x^i) \tag{3.12}$$

where $f(x^i)$, $h(x^i)$, and $l^\mu(x^i)$ are some functions of their arguments. Then substituting (3.12) into (3.11c), we get

$$\partial_\mu f = 0, \qquad \partial_\mu h = 0 \tag{3.13a}$$

$$l^\mu \partial_\mu \ln \sqrt{\gamma} + \partial_\mu l^\mu = 0 \tag{3.13b}$$

Finally,

$$\eta^\mu = f(x^4) D^\mu + h(x^4) B^\mu + l^\mu(x^i) \tag{3.14}$$

with $l^\mu(x^i)$ satisfying equation (3.13b).

Analogously, the invariance condition (3.7) applied to the second equation of (2.20b) gives

$$\xi^\mu = m(x^4) D^\mu + p(x^4) B^\mu + s^\mu(x^i) \tag{3.15}$$

where $m(x^4)$, $p(x^4)$, and $s^\mu(x^i)$ are some functions of their arguments and

$$s^\mu \partial_\mu \ln \sqrt{\gamma} + \partial_\mu s^\mu = 0 \tag{3.16}$$

The invariance condition for the Einstein equations (2.4) yields

$$X T^i{}_j = 0 \tag{3.17}$$

It is a simple matter to show that from (3.17) under (3.2), (3.14), and (3.15) one gets

$$XM = 0 \qquad (\Leftrightarrow M \text{ is invariant under } G_1) \tag{3.18a}$$

$$h + m = 0, \qquad l^\mu = 0, \qquad s^\mu = 0 \tag{3.18b}$$

Finally, straightforward but rather long calculations show that the invariance condition (3.7) with (3.14), (3.15), (3.18a), and (3.18b) when applied to the first equations of (2.20a) and (2.20b) give

$$D^\mu \partial_4 f - B^\mu \partial_4 m = 0, \qquad D^\mu \partial_4 m + B^\mu \partial_4 p = 0 \tag{3.19}$$

i.e.,

$$m(x^4) = a, \qquad p(x^4) = b, \qquad f(x^4) = c, \qquad a, b, c \subset \mathbb{R} \tag{3.20}$$

Collecting these results, we find that

$$X = (aD^\mu + bB^\mu) \frac{\partial}{\partial D^\mu} + (cD^\mu - aB^\mu) \frac{\partial}{\partial B^\mu}$$
$$\begin{Vmatrix} a & b \\ c & -a \end{Vmatrix} \in sl(2; \mathbb{R}) \tag{3.21}$$

where $sl(2; \mathbb{R})$ is the Lie algebra of the Lie group $SL(2; \mathbb{R})$. Thus one arrives at the following important result:

Theorem 3.1. Let G_r be an r-parameter local Lie group of transformations of the form

$$\begin{gathered} x'^i = x^i, \qquad g'_{ij} = g_{ij} \\ \mathbf{D}' = \mathbf{D}'(x^i, \mathbf{D}, \mathbf{B}; \tau_1, \ldots, \tau_r), \qquad \mathbf{B}' = \mathbf{B}'(x_i, \mathbf{D}, \mathbf{B}; \tau_1, \ldots, \tau_r) \end{gathered} \tag{3.22}$$

where $\tau_1, \ldots, \tau_r \in \mathbb{R}$ are the group parameters.

Then the Einstein–Maxwell equations (2.4), (2.20a), and (2.20b) are invariant under G_r iff:

(i) G_r is a local subgroup of the group $SL(2; \mathbb{R})$,

$$\begin{gathered} x'^i = x^i, \qquad g'_{ij} = g_{ij} \\ \mathbf{D}' = a_{11}\mathbf{D} + a_{12}\mathbf{B}, \qquad \mathbf{B}' = a_{21}\mathbf{D} + a_{22}\mathbf{B} \\ \det \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix} = 1 \end{gathered} \tag{3.23}$$

(ii) M is an invariant of G_r.

Remark. We deal only with the connected groups.

It is evident that our theorem has been proved in some fixed coordinate system. Now a crucial point is whether the invariant group G_r is independent of the coordinate system. To answer this question we consider the invariance condition for M, (3.18a), in the form [see (2.18)]

$$X[\mathbf{B} \cdot \mathbf{H} - \sqrt{-g_{44}} K(P, Q)] = 0 \tag{3.24}$$

Then by simple but long calculations, using (2.13) and (2.14), one finds that the condition (3.24) for X given by (3.21) reads

$$2a\left(\frac{\partial K}{\partial P} P + \frac{\partial K}{\partial Q} Q\right) + ib\left\{\left[\left(\frac{\partial K}{\partial P}\right)^2 + \left(\frac{\partial K}{\partial Q}\right)^2\right] Q + 2 \frac{\partial K}{\partial P} \frac{\partial K}{\partial Q} P\right\} + icQ = 0 \tag{3.25}$$

Concluding, we can state that M is an invariant function for G_r iff the condition (3.25) holds for each infinitesimal operator of G_r. Now we want this condition to be satisfied in an arbitrary coordinate system. As K and P are scalars and Q is a pseudoscalar, the condition (3.25) is satisfied in an arbitrary coordinate system iff [compare with Białynicki-Birula (1983)]

$$b\left\{\left[\left(\frac{\partial K}{\partial P}\right)^2+\left(\frac{\partial K}{\partial Q}\right)^2\right]Q+2\frac{\partial K}{\partial P}\frac{\partial K}{\partial Q}P\right\}+cQ=0 \tag{3.26a}$$

and

$$a\left(\frac{\partial K}{\partial P}P+\frac{\partial K}{\partial Q}Q\right)=0 \tag{3.26b}$$

Note that using (2.1), (2.2), (2.7), and (2.15), one can write (3.26a) and (3.26b) in the following form:

$$bG+cQ=0 \tag{3.27a}$$

$$a(K-L)=0 \qquad \text{or} \qquad a(\mathbf{B}\cdot\mathbf{H}-\mathbf{E}\cdot\mathbf{D})=0 \tag{3.27b}$$

Then we arrive at the following theorem:

Theorem 3.2. Let G_r be an r-parameter local Lie group of transformations defined by (3.22).

Then the Einstein–Maxwell equations (2.4), (2.20a), and (2.20b) are invariant under G_r in arbitrary coordinate system iff:

(i) G_r is a local subgroup of $SL(2;\mathbb{R})$ of the form (3.23).

(ii) The conditions (3.26a), (3.26b) [or (3.27a), (3.27b)] are satisfied for each element of the Lie algebra g_r of G_r,

$$\left\|\begin{matrix} a & b \\ c & -a \end{matrix}\right\| \in g_r$$

The maximal local group which Theorem 3.2 deals with we call the *group of duality transformations* and we denote it by DT. One easily shows that in terms of f_{ij} and p_{ij} the general duality transformation (3.23) can be written as follows:

$$\begin{aligned} f'_{ij} &= a_{22}f_{ij}-ia_{21}\,{}^*p_{ij} \\ {}^*p'_{ij} &= ia_{12}f_{ij}+a_{11}\,{}^*p_{ij} \end{aligned} \tag{3.28}$$

4. ELECTRODYNAMICS ADMITTING THE DUALITY TRANSFORMATION GROUP

In the preceding section it has been shown (see Theorems 3.1 and 3.2) that $XM=0$ in an arbitrary coordinate system iff (3.26a) and (3.26b), or equivalently equations (3.27a) and (3.27b), hold. Consider an algebraically general electromagnetic field, i.e.,

$$F + G \neq 0 \tag{4.1}$$

By (2.2), (2.7), and (2.15) we have

$$F + G \neq 0 \Leftrightarrow P + Q \neq 0 \tag{4.2}$$

It is well known (Plebański, 1974; Salazar *et al.*, 1987) that if $P + Q \neq 0$, then one can choose a coordinate system in such a manner that at some point of M_4

$$\|g_{ij}\| = \text{diag}\|1, 1, 1, -1\|, \qquad \mathbf{D} = (0, 0, D), \qquad \mathbf{H} = (0, 0, H), \qquad D, H \in \mathbb{R} \tag{4.3}$$

Then by (2.14)

$$P = \frac{1}{2}(H^2 - D^2), \qquad Q = iDH \tag{4.4}$$

From (2.10a), (2.10b), (2.7), and (4.3) one finds

$$\mathbf{E} = (0, 0, E), \qquad \mathbf{B} = (0, 0, B), \qquad E, B \in \mathbb{R} \tag{4.5}$$

and consequently by (2.15)

$$F = \frac{1}{2}(B^2 - E^2), \qquad G = iEB \tag{4.6}$$

Now M can be considered to be a function of D and B, $M = M(D, B)$, and by (2.19) we have

$$E = \frac{\partial M}{\partial D}, \qquad H = \frac{\partial M}{\partial B} \tag{4.7}$$

Finally, using (3.27a), (3.27b), and (4.3)–(4.7) one finds that equations (3.26a) and (3.26b) read

$$bB \frac{\partial M}{\partial D} + cD \frac{\partial M}{\partial B} = 0 \tag{4.8a}$$

$$a\left(B \frac{\partial M}{\partial B} - D \frac{\partial M}{\partial D}\right) = 0 \tag{4.8b}$$

Simple analysis shows that equations (4.8a) and (4.8b) give two essentially distinct solutions, which, without any loss of generality, can be written as follows:

$$a = 1, \qquad b = 0 = c; \qquad M = M(D \cdot B) \tag{4.9}$$

or

$$a = 0, \qquad b = 1, \qquad c \neq 0; \qquad M = M(\tfrac{1}{2}(B^2 - cD^2)) \tag{4.10}$$

It is evident that in the case described by (4.9) if $D=0$, then $H=0$ and this contradicts the assumption that **D** and **H** are independent variables. Thus we are left only with the case described by (4.10). Moreover, from (4.10) it follows that any DT group appears to be a one-parameter Lie group. Given M of the form (4.10), one finds B from (4.7) to be a function of D and H, $B=B(D, H)$. Then using (2.18) with $g_{44}=-1$ one gets K as a function of D and H,

$$K = HB(D, H) - M(D, B(D, H)) \tag{4.11}$$

Then from (4.4) we obtain (changing eventually coordinates of the 3-space)

$$D = [(P^2 - Q^2)^{1/2} - P]^{1/2}, \qquad H = [(P^2 - Q^2)^{1/2} + P]^{1/2} \tag{4.12}$$

Substituting (4.12) into (4.11), one gets the structural function $K = K(P, Q)$ which satisfies equations (3.26a) and (3.26b).

Now we are able to find the general element of any DT group. From (4.10) it follows (Plebański and Przanowski, 1988*a*) that for a fixed c any element of the DT group reads

$$\exp\left\{\tau \begin{Vmatrix} 0 & 1 \\ c & 0 \end{Vmatrix}\right\} = \begin{Vmatrix} \cosh(\sqrt{c}\tau) & \dfrac{\sinh(\sqrt{c}\tau)}{\sqrt{c}} \\ c\dfrac{\sinh(\sqrt{c}\tau)}{\sqrt{c}} & \cosh(\sqrt{c}\tau) \end{Vmatrix} \in DT, \qquad \tau \in \mathbb{R} \tag{4.13}$$

Remark. In our previous paper (Plebański and Przanowski, 1988*a*) there is an error in the definition of the scalar product $(v|v)$. The correct definition reads $(v|v) = +K_{ij}v^iv^j$. Consequently, formula (2.25) of that paper takes the form $z = (x + y + x \wedge y)/[1 + (x|y)]$.

Here is the place to examine the model of electrodynamics which admits the DT group from the physical viewpoint.

First, one usually assumes that the dominant energy condition holds, i.e., for every nonvanishing electromagnetic field and for every timelike vector u^i, $u^iu_i < 0$,

$$T_{ij}u^iu^j > 0 \text{ and the vector } T_{ij}u^j \text{ is nonspacelike} \tag{4.14}$$

[For the energy conditions see Hawking and Ellis (1973) and Guzmán-Sánchez *et al.* (1991).]

It is an easy matter to show (Salazar *et al.*, 1987; Plebański and Przanowski, 1992) that with M given by (4.10) the dominant energy conditions (4.14) are satisfied iff

$$M \geq y\frac{dM}{dy} > 0 \qquad \text{for} \quad (D^2 + B^2) \neq 0 \tag{4.15}$$

where $y := \frac{1}{2}(B^2 - cD^2)$.

Then we find that the condition (4.15) can be satisfied only if $c<0$. Therefore taking in (4.13) $\sqrt{c}\tau = i\varphi$, $\varphi \in \mathbb{R}$, and $\sqrt{c} = i\sqrt{-c}$ one gets the group which can be called the *generalized duality rotation group GDR*,

$$GDR = \left\{ \left\| \begin{matrix} \cos\varphi & \dfrac{\sin\varphi}{\sqrt{-c}} \\ -\sqrt{-c}\sin\varphi & \cos\varphi \end{matrix} \right\|, \varphi \in \mathbb{R} \right\} \tag{4.16}$$

Second, it is also physically reasonable to assume that an electrodynamics corresponds to the Maxwell electrodynamics for weak fields. This means that $M(D, B)$ is of the form

$$M(D,B) = \tfrac{1}{2}(D^2+B^2) + o(D^2+B^2) \tag{4.17}$$

Comparing this with (4.15), we have $c = -1$ and then one finds $M = M(D, B)$ to be

$$\begin{aligned} &M = M\left(\frac{1}{2}(D^2+B^2)\right), \qquad M(0)=0, \qquad \lim_{y\to 0}\frac{dM}{dy}=1 \\ &M \geq y\frac{dM}{dy} > 0 \qquad \text{for} \quad y>0; \qquad y := \frac{1}{2}(D^2+B^2) \end{aligned} \tag{4.18}$$

Moreover, with $c = -1$ the generalized duality rotation group appears to be the *duality rotation group DR*,

$$DR = \left\{ \left\| \begin{matrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{matrix} \right\|, \varphi \in \mathbb{R} \right\} \tag{4.19}$$

Therefore the transformation of **D** and **B** is given by (1.3), which implies also (1.2). In terms of f_{ij} and p_{ij} the duality rotation can be written as follows [see (3.28)]:

$$\begin{aligned} f'_{ij} &= f_{ij}\cos\varphi + i\,{}^*p_{ij}\sin\varphi \\ {}^*p'_{ij} &= {}^*p_{ij}\cos\varphi + if_{ij}\sin\varphi \end{aligned} \tag{4.20}$$

Collecting results, one arrives at the following fundamental theorem:

Theorem 4.1. Let M satisfy the conditions (4.18). Then it generates an electrodynamics which:

(i) Admits a *DT* group.
(ii) Satisfies the dominant energy condition (4.14).
(iii) Corresponds to the Maxwell electrodynamics for weak fields.

Moreover, the *DT* group appears now to be the *DR* group. Conversely, any electrodynamics satisfying the conditions (i)–(iii) is generated by M of the form (4.18).

This result was found by Salazar *et al.* (1987). Here we have proved it in the whole generality.

Examples. (A) Maxwell electrodynamics. Here

$$M = \frac{1}{2}(D^2 + B^2) \tag{4.21}$$

Then

$$E = D, \qquad H = B; \qquad K = P, f_{ij} = p_{ij}, L = -F \tag{4.22}$$

(B) Born–Infeld nonlinear electrodynamics (Born and Infeld, 1934; Plebański, 1968; Białynicki-Birula and Białynicka-Birula, 1975; Salazar *et al.*, 1987, 1989). Now we put

$$M = b^2\{[1 + b^{-2}(D^2 + B^2)]^{1/2} - 1\}, \qquad 0 \neq b \in \mathbb{R} \tag{4.23}$$

Then by (4.4), (4.7), and (4.11) one gets

$$K = b^2 - [(b^2 + D^2) \cdot (b^2 - H^2)]^{1/2} = b^2 - (b^4 - 2b^2P + Q^2)^{1/2} \tag{4.24}$$

Finally, as

$$L = DE - M \tag{4.25}$$

(4.6), (4.7), and (4.25) yield

$$L = b^2 - [(b^2 - E^2) \cdot (b^2 + B^2)]^{1/2} = b^2 - (b^4 + 2b^2F + G^2)^{1/2} \tag{4.6}$$

5. AN EXTENSION OF THE RAINICH–MISNER–WHEELER THEORY

The famous "already unified field theory" of Rainich, Misner and Wheeler gives, in terms of the space-time geometry, necessary and sufficient conditions for a space-time to admit the Maxwell algebraically general electromagnetic field without sources (Rainich, 1925; Misner and Wheeler, 1957; Witten, 1962; Przanowski, 1983; Hammon, 1990). The crucial point in this theory appears to be the fact that the Maxwell electrodynamics admits the duality rotation group (compare also with Deser and Teitelboim, 1976). Therefore the natural question arises whether one can extend the classical Rainich–Misner–Wheeler theory to the case of any electrodynamics satisfying the conditions (i)–(iii) of Theorem 4.1. This problem was solved in our previous work (Plebański and Przanowski, 1992). In the present paper we give a slightly improved version of that work. [For an application of the extended Rainich–Misner–Wheeler theory see Salazar *et al.* (1987).]

First, using the results of our previous considerations, one can prove the following theorem:

Theorem 5.1. Let a space-time geometry satisfy the following conditions: (a)

$$C^{ij}C_{jk} = \frac{1}{4} C^{jl}C_{jl}\delta^{i}{}_{k}, \qquad C^{ij}C_{ij} \neq 0 \tag{5.1}$$

where $C_{ij} := R_{ij} - \frac{1}{4}Rg_{ij}$.

(b) The curvature scalar R is a function of $I := (C^{ij}C_{ij})^{1/2}$, $R = R(I)$, such that

$$\frac{dR}{dI} - 2 \neq 0 \tag{5.2}$$

(c) $C_{ij}v^i v^j < 0$ for some nonspacelike vector v^i,

$$R + 4\Lambda \leq 0 \qquad \text{with} \quad \Lambda := -\frac{1}{4} \lim_{I \to 0} R(I) \tag{5.3}$$

Then the function $M = M(y)$ defined by the relations

$$M = \frac{1}{8}(2I - R - 4\Lambda) \tag{5.4a}$$

$$y = I \exp\left(-\frac{1}{2}\int I^{-1}\frac{dR}{dI}\,dI\right) \tag{5.4b}$$

where the integral constant is taken so to give

$$\lim_{y \to 0} \frac{dM}{dy} = 1 \tag{5.4c}$$

defines an algebraically general "electromagnetic field" according to the procedure described in the preceding sections with $y = \frac{1}{2}(D^2 + B^2)$. This field satisfies the Einstein equations (2.4) and the dominant energy condition (4.14), and it is defined with accuracy to the duality rotation (4.20) with φ being now an arbitrary function on the space-time. Moreover, the "electrodynamics" generated by the M corresponds to the "Maxwell electrodynamics" for weak fields.

We use quotation marks because Theorem 5.1 does not decide if the Maxwell equations (2.6) are satisfied.

Examples. (A) Let

$$R(I) = -4\Lambda \tag{5.5}$$

Then by (5.4b)

$$y = cI, \qquad 0 < c = \text{const} \tag{5.6}$$

Consequently (5.4a) gives

$$M = \frac{1}{4c} y \tag{5.7}$$

From (5.4c) and (5.7) one gets $c = 1/4$, and finally

$$M = y, \qquad y = \frac{1}{2}(D^2 + B^2) \tag{5.8}$$

This is of course the case of the Maxwell electrodynamics.

(B) Assume that

$$R(I) = -4\Lambda - 2[(I^2 + c)^{1/2} - \sqrt{c}], \qquad 0 < c = \text{const} \tag{5.9}$$

Then one finds M to be

$$M = b^2[(1 + 2b^{-2}y)^{1/2} - 1], \qquad b := \frac{1}{2} c^{1/4} \tag{5.10}$$

This M [with $y = \frac{1}{2}(D^2 + B^2)$] generates the Born–Infeld nonlinear electrodynamics.

Now, as is done in the classical Rainich–Misner–Wheeler theory, we should find the conditions which assure that the Maxwell equations (2.6) are satisfied. First, it is well known (Plebański, 1974) that if the conditions (5.1) are satisfied, then there exists a null tetrad (e^1, e^2, e^3, e^4) such that

$$C_{ab} e^a \otimes e^b = \frac{1}{2} I(e^3 \otimes e^4 + e^4 \otimes e^3 - e^1 \otimes e^2 - e^2 \otimes e^1)$$
$$a, b = 1, \ldots, 4 \tag{5.11}$$

[A null tetrad is defined to be four 1-forms (e^1, e^2, e^3, e^4), $\overline{e^1} = e^2$, $\overline{e^3} = e^3$, and $\overline{e^4} = e^4$ (where the bar stands for the complex conjugation) such that the space-time metrix ds^2 reads

$$ds^2 = g_{ab} e^a \otimes e^b, \qquad \|g_{ab}\| = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{Vmatrix} \tag{5.12}$$

For details see Plebański (1974) and Plebański and Przanowski (1988*b*).] Define the following 2-form:

$$\xi := e^1 \wedge e^2 = \frac{1}{2} \xi_{ij} dx^i \wedge dx^j \tag{5.13}$$

By the virtue of (5.11) the 2-form ξ is defined within to the sign. Define the 1-form α by

$$\alpha := -i\left(\frac{I}{4y}\xi^{ij}{}_{;j}\,{}^*\xi_{ik} + \frac{4y}{I}\,{}^*\xi^{ij}{}_{;j}\xi_{ik}\right) dx^k \tag{5.14}$$

where y is defined according to (5.4).

Note that in the case of the Maxwell electrodynamics

$$\begin{aligned}\alpha &= -i(\xi^{ij}{}_{;j}\,{}^*\xi_{ik} + {}^*\xi^{ij}{}_{;j}\xi_{ik})\, dx^k \\ &= -\sqrt{-g}\epsilon_{ijlk}\frac{C^i{}_m C^{mj;l}}{I^2}\, dx^k\end{aligned} \tag{5.15}$$

With all this, one can prove the following theorem:

Theorem 5.2. Let the assumptions of Theorem 5.1 be satisfied and let

$$d\alpha = 0 \tag{5.16}$$

Then the space-time admits the existence of an electromagnetic field such that the Einstein–Maxwell equations are fulfilled. The electromagnetic 2-form is given by

$$\begin{aligned}\omega &:= \frac{1}{2}(f_{ij} + {}^*p_{ij})\, dx^i \wedge dx^j \\ &= \frac{1}{2}(2y)^{1/2} e^{i(\phi+\phi_0)}\left(\xi_{ij} + \frac{dM}{dy}\,{}^*\xi_{ij}\right) dx^i \wedge dx^j\end{aligned} \tag{5.17}$$

where y and $M(y)$ are defined as in Theorem 5.1; ϕ is any solution of the equation

$$\alpha = d\phi \tag{5.18}$$

and ϕ_0 is an arbitrary real constant.

It is evident that any arbitrary real parameter ϕ_0 can be identified with the parameter φ of the duality rotation group. Theorems 5.1 and 5.2 generalize the analogous results of the classical Rainich–Misner–Wheeler theory.

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